4.5 Algebraic solution by reduction to polynomial

Transcendental equations are difficult to solve because, although there may be just one variable, say θ, it generally appears as $\sin\theta$ and $\cos\theta$. Making the following substitutions, however, yields an expression in terms of a single variable, u:

$$
\begin{aligned}
u &= \tan\frac{\theta}{2}, \\
\cos\theta &= \frac{1-u^2}{1+u^2}, \\
\sin\theta &= \frac{2u}{1+u^2}.
\end{aligned}
\tag{4.35}
$$

This is a very important geometric substitution used often in solving kinematic equations. Using these substitutions, transcendental equations are converted into polynomial equations in u. Appendix A lists these and other trigonometric identities.

EXAMPLE 4.3

Convert the transcendental equation

$$a\cos\theta + b\sin\theta = c, \tag{4.36}$$

into a polynomial in the tangent of the half angle and solve for θ.

Substituting from (4.35) and multiplying through by $1+u^2$ we have

$$a(1-u^2) + 2bu = c(1+u^2). \tag{4.37}$$

Collecting powers of u yields

$$(a+c)u^2 - 2bu + (c-a) = 0, \tag{4.38}$$

which is solved by the quadratic formula to be

$$u = \frac{b \pm \sqrt{b^2 - a^2 - c^2}}{a+c}. \tag{4.39}$$

Hence,

$$\theta = 2\tan^{-1}\left(\frac{b \pm \sqrt{b^2 - a^2 - c^2}}{a+c}\right). \tag{4.40}$$

Should the solution for u from (4.39) be complex, there is no real solution to the original transcendental equation. Note that if $a + c = 0$ the argument of the arctangent becomes infinity, and hence, $\theta = 180°$. In a computer implementation this potential division by zero should be checked for ahead of time. This situation results from the quadratic term of (4.38) vanishing so that the quadratic degenerates into a linear equation. ■

APPENDIX C

Some inverse-kinematic formulas

The single equation

$$\sin\theta = a \tag{C.1}$$

has two solutions, given by

$$\theta = \pm\text{Atan2}(\sqrt{1-a^2}, a). \tag{C.2}$$

Likewise, given

$$\cos\theta = b, \tag{C.3}$$

there are two solutions:

$$\theta = \text{Atan2}(b, \pm\sqrt{1-b^2}). \tag{C.4}$$

If both (C.1) and (C.3) are given, then there is a unique solution given by

$$\theta = \text{Atan2}(a, b). \tag{C.5}$$

The transcendental equation

$$a\cos\theta + b\sin\theta = 0 \tag{C.6}$$

has the two solutions

$$\theta = \text{Atan2}(a, -b) \tag{C.7}$$

and

$$\theta = \text{Atan2}(-a, b). \tag{C.8}$$

The equation

$$a\cos\theta + b\sin\theta = c, \tag{C.9}$$

which we solved in Section 4.5 with the tangent-of-the-half-angle substitutions, is also solved by

$$\theta = \text{Atan2}(b, a) \pm \text{Atan2}(\sqrt{a^2+b^2-c^2}, c). \tag{C.10}$$

The set of equations

$$\begin{aligned} a\cos\theta - b\sin\theta &= c, \\ a\sin\theta + b\cos\theta &= d, \end{aligned} \tag{C.11}$$

which was solved in Section 4.4, also is solved by

$$\theta = \text{Atan2}(ad - bc,\ ac + bd). \tag{C.12}$$

APPENDIX A

Trigonometric identities

Formulas for rotation about the principal axes by θ:

$$R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \tag{A.1}$$

$$R_Y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \tag{A.2}$$

$$R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{A.3}$$

Identities having to do with the periodic nature of sine and cosine:

$$\begin{aligned} \sin\theta &= -\sin(-\theta) = -\cos(\theta + 90°) = \cos(\theta - 90°), \\ \cos\theta &= \cos(-\theta) = \sin(\theta + 90°) = -\sin(\theta - 90°). \end{aligned} \tag{A.4}$$

The sine and cosine for the sum or difference of angles θ_1 and θ_2:

$$\begin{aligned} \cos(\theta_1 + \theta_2) &= c_{12} = c_1 c_2 - s_1 s_2, \\ \sin(\theta_1 + \theta_2) &= s_{12} = c_1 s_2 + s_1 c_2, \\ \cos(\theta_1 - \theta_2) &= c_1 c_2 + s_1 s_2, \\ \sin(\theta_1 - \theta_2) &= s_1 c_2 - c_1 s_2. \end{aligned} \tag{A.5}$$

The sum of the squares of the sine and cosine of the same angle is unity:

$$c^2\theta + s^2\theta = 1. \tag{A.6}$$

If a triangle's angles are labeled a, b, and c, where angle a is opposite side A, and so on, then the "law of cosines" is

$$A^2 = B^2 + C^2 - 2BC\cos a. \tag{A.7}$$

The "tangent of the half angle" substitution:

$$\begin{aligned} u &= \tan\frac{\theta}{2}, \\ \cos\theta &= \frac{1 - u^2}{1 + u^2}, \\ \sin\theta &= \frac{2u}{1 + u^2}. \end{aligned} \tag{A.8}$$

To rotate a vector Q about a unit vector $\hat{K}$ by θ, use **Rodriques's formula**:

$$Q' = Q\cos\theta + \sin\theta(\hat{K} \times Q) + (1 - \cos\theta)(\hat{K} \cdot \hat{Q})\hat{K}. \tag{A.9}$$

See Appendix B for equivalent rotation matrices for the 24 angle-set conventions and Appendix C for some inverse-kinematic identities.